

C H R I S T I A N I H U G E N I I

L I B E L L U S

D E

R A T I O C I N I I S

I N

*L U D O A L E Æ .*

---

O R,

The V A L U E of all

**C H A N C E S**

I N

**Games of Fortune;**

C A R D S, D I C E, W A G E R S,  
L O T T E R I E S, &c.

Mathematically Demonstrated.

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*L O N D O N :*

Printed by S. KEIMER for T. WOODWARD, near  
the *Inner Temple-Gate* in *Fleetstreet*. 1714.

[This document consists of a reprint of an English translation of the first book on probability theory ever published, viz. Christiaan Huygens' *De Ratiociniis in Ludo Aleae* which was published in Latin in 1657. The reprint of the translation is complete. The original pagination is indicated in square brackets.] [d1]

T O

Dr. *RICHARD MEAD*,

Physician to St. *Thomas's*  
Hospital and Fellow of  
the ROYAL SOCIETY.

*HONOURED SIR,*

**W**HEN I consider the Subject of the following papers, I can no more forbear dedicating them to Your Name, than I can refuse giving my assent to any Proposition in these Sciences, which I have already seen clearly demonstrated. The Reason is plain, for as You have contributed to the greatest Lustre and Glory to an very considerable part of the Mathematicks, by introducing them into their noblest Province, the Theory of Physick ; the Publisher of any Truths of that Nature, who is desirous of seeing them come to their utmost Perfection, must of course beg Your Patronage and Application of them. By so prudent a Course as this, he may perhaps see those Propositions which [d2] it was his utmost Ambition to make capable only of directing Men in the Management of their Purses, and instructing them to what Chances and Hazards they might safely commit their Money ; turn'd some time or other to a much more glorious End, and made instrumental likewise towards the securing their Bodies from the Tricks of that too successful Sharper, Death, and countermining the underhand Dealings of secret and over-reaching Distempers.

THE most celebrated Endeavours of the greatest GENIUS that ever appear'd before in Mathematical Learning, have been able to carry it no farther, than to calculate by its means the Motions of the Heavenly Bodies more exactly, explain the Mystery of Tides, The Doctrine of Sounds and Light, and other curious Phœnomena in the Works of Nature, which are without us: But You, Great Sir, have made this wonderful Clew conduct us in Paths much more intricate and obscure, and guide us thro' the far more more curious and puzzling Labyrinths of Human Body ; and have as far out-done those noblest improvements, as the glorious Frame of Man exceeds all the rest of Nature's Works, and as the Knowledge of ourselves is preferable to all other parts of

Science whatever. All our Hopes that this Sort of Learning will meet with a warmer and more welcome Reception from the present [d3] and future Ages, than it has from those past, are entirely owing to Your instructing us in the true Value of Mathematical Disquisitions, and discovering to us their most excellent Uses which our Ancestors were perfectly ignorant of. The Mathematicks being formerly believ'd of no further Service, than as they assisted us in the Business of our Hands, by directing our Mechanical Operations, or diverted our Heads with pleasant and amusing Problems, where then but particular Studies, and cultivated only by some few as a necessary help to their employments, and by fewer as a Diversion ; but since You, to our extreme Surprize and Satisfaction, have shown their scarce dream'd of Excellencies in promoting the Profession of Physick almost to a pitch of Certainty, and demonstrated them capable, by a due Application of adding Days and Years to the Length of our Lives, they must now necessarily become the general Concern of all Mankind and the greatest Subject of their Admiration and Esteem.

PERHAPS the few following Propositions, may to the World appear so foreign to this Purpose, as to be thought perfectly incapable of being directed to such a mighty End. For what can there be in common between the Value of a Chance in a Game, and the Knowledge and Cure of a Distemper? And how can the nicest Determination of the former, any way influence [d4] or illustrate the latter? But these apparent Inconsistencies, Sir, are not such as can impose upon Your clearer Judgement, at the Light of which they vanish like Phantoms at the break of Day. You have reconciled Truths that have seem'd to us much more inconsistent than these, and even demonstrated what the wiser part of Mankind before, had always look'd upon as absurd and impossible, That the State of our Bodies on Earth, are subject to surprizing Alterations and Changes from the various Positions of those two heavenly ones, the SUN and MOON. So that we can now talk of their Conjunctions, Oppositions, and Quadratures, in relation to Human Diseases, without being any longer liable to the usual Reproaches of Ignorance and Infatuation, justly chargeable upon the old judicial Astrology. Your late learned Friend, Dr. PITCAIRNE, made no difficulty of lifting this Subject into the Service of Physick, when he gave our Third Proposition a very honourable Post in his Dissertat. de Circulatione sanguinis, and found it of Force enough to overthrow and defeat for ever, that erroneous and long standing Doctrine of Secretion, which supposes the Orifice of each Secretory Duct to be of a different Figure, and as admitting only those Particles of the Blood that are of a similar Figure and Magnitude. Yes, there is but too much Chance and [d5] Uncertainty in Human Constitutions, and the Duration of this mortal Life ; and if a proper Application of this Doctrine might assist us to get a

*true and exact Value upon that Uncertainty, perhaps the Physicians will not be the only Persons that are like to benefit by it ; but the Divines likewise may be furnish'd from thence with the most convincing and, I think, only Arguments they have left to make use of ; for that eminent Body have so far improv'd in their Admonitions and Reasonings with Mankind, that nothing but Mathematical Demonstration seems able to do them any further Service. Such an Application as this, together with all the Advantages the other Parts of the Mathematicks can confer upon the Science of Physick, are what we must only expect from Your Pen ; whenever Your Hurry or Business will allow You a few leisure Intervals, and a convenient Opportunity to resume it.*

*YOU have prophesy'd, Sir, in the Preface to Your excellent Piece upon Poisons, that the true Physician shall hereafter be distinguish'd from the QUACK, not only by a perfect acquaintance with the learned Languages, but a considerable Proficiency likewise in the Mathematicks: and the Prognostick is pretty surely grounded; for that happy Distinction is not only foretold, but establish'd by your Writings, the Pleasure of reading which, will most certainly compleat that [d6] Prediction . For the Veneration the World pays to Your Works, is so equal to their Value, that we find none desirous of being accounted Physicians, who have not first made themselves thoroughly acquainted with them ; and consequently none are like to be so, without the Qualifications by You precribed.*

*I have made no mention, Sir, of the many Personal Obligations, I lie under to You, and the singular Favours and Encouragement I have met with at Your Hands, because they cannot be imagin'd to have been any Motives to this Address ; which is so far from being able to discharge the smallest of that great Debt, that on the part of Mr. HUYGENS, it is only my Duty, and but Justice done You, and on my own, a very considerable Addition to what I already owe Your unlimited Goodness and Generosity, and too great an Honour done to,*

SIR,

Your most Obedient,

Obliged, and humble Servant,

W. Browne.

[a1]  
 ADVERTISEMENT  
 TO THE  
 READER

**A**S alle Mathematical Studies in general are unaccountably bewitching and delightful to those that are once hapily engaged in tme ; so that part which considers and estimates our *Expectations of Events* that are themselves *uncertain*, and depend intirely upon *Chance* and *Hazard*, cannot fail of giving a particular Pleasure and Satisfaction. To reduce the unconstant and irregular Proceedings of blind *Fortune* to certain Rules and Limits, and to set a definite *Value* upon her capricious *Favours* and *Smiles*, seems to be Undertakings of so chimærical a Nature, that there is no Body but must be delightfully surprized with that Art which discovers them both really possible, and with a little Application easily practicable

But I think there is little need of saying in Commendation either of the Subject, any thing or of the Manner in which has consider'd it ; the great scarcity of his little [a2] Treatise upon it, and the general want of a new Edition making that altogether superfluous. The just value every one has for his Performance this Way, as well as for his other admirable Pieces, abundantly show that they are sufficiently sensible of his Excellencies; and is the greatest Commendation it can possibly receive. Besides the Latin Editions it has passed thro', the learned Dr. ARBUTHNOTT publish'd an *English* one, together with an Application of the General Doctrine to some particular *Games* then most in use; which is so entirely dispers'd Abroad, that an Account of it is all we can now meet with. A late *French Author* has indeed illustrated this Work with a plentiful Number of Instances where it may be servicable ; but all that he has done serves only to demonstrate what an inexhaustible Treasure this little Book of M. HUYGENS contains, and how easily it will satisfy every ones Occasions, that will but be at the Pains to make use of it. Our excellent Analyst M. DE MOIVRE likewise had wonderfully improv'd the Subject, and besides taking the different Dexterities of the Gamesters into the Account, has much shorten'd the Calculation, and made the whole more general and compleat. But nothing can reflect more Honour upon M. HUYGENS than this last admirable Performance, which must needs show us what an excellent Foundation that learned Man laid, that cou'd support and give Strenght to such a glorious Structure as M. DE MOIVRE has rais'd upon it. M. DE MOIVRE'S Piece therefore will be far from lessening the Worth of M. HUYGENS'S [a3] and the superficial Mathematicians will still be glad to satisfy their enquiries by this last Author's easy tho' more tedious Method, as not being able to understand the other's more comprehensive and general

one; while those of a greater depth, will with no less delight first read M. HUYGENS'S Treatise, in order to proceed with so much the greater Pleasure afterwards to peruse M. DE MOIVRE'S Additions, and to perceive by what just Steps and Degrees the Subject has from these two Great Men receiv'd both its first Consideration and last Perfection.

My Design in publishing this Edition, was to have made it as Useful as possible, by an Addition of a very large *Appendix* to it, containing a Solution of some of the most servicable and intricate Problems I cou'd think of, and such as have not as yet, that I know of, met with a particular Consideration : But an Information I have within these few Days received, that M. MONMORT'S *French* Piece is just newly reprinted at *Paris* with very considerable Additions, has made me put a Stop to the *Appendix*, till I can procure a Sight of what has been added anew, for fera some part of it may possibly have been honour'd with the Notice and Consideration of that ingenious Author. If the Reader is of Opinion that I shou'd likewise till then, have deferr'd the Publication of these few Pages, tho' printed off ; and thinks much that he has not either all that was design'd hem or none at all; or if he has any Exceptions to their appearing in the *English* Language, the Bookseller is the only Person that is to answer for his not receiving Satisfaction in both those Points.

*Just Published the following BOOKS, Printed for and Sold by THO. WOODWARD, next the Inner Temple-Gate in Fleetstreet.*

A Short and easy Method to understand GEOGRAPHY, wherein are described the Form of Government of each Country, its Qualities, the Manners of its Inhabitants, and whatsoever is most Remarkable in it. To which are added, Observations upon those Things of Importance that have happen'd in each State: With an ABRIDGEMENT of the *Sphere*, and the Description and Use of the *Globe*, *Geographical Maps* and *Sea-Charts*, Englished by a Gentleman of *Cambridge*, from the *French* of Monsieur A. D. Fer, Geographer to the *French* King.

The Seventh Edition of an Help and Exhortation to *Worthy Communicating* ; or, a Treatise describing the Meaning, worthy Reception, Duty and Benefits of the *Holy Sacrament*; And answering the Doubts of Conscience, and other Reasons, which most generally detain Men from it. Together with suitable Devotions added. By JOHN KETTLEWELL, late Vicar of *Coles-hill* in *Warwickshire*.

The Fifth Edition of a SERMON preach'd to the *Protestants of Ireland*, now in London, at the Parish-Church of St. *Mary-le Bow*, Oct.23. 1714. Being the Day appointed by Act of Parliament in *Ireland*, for an *Anniversary Thanksgiving* for the Deliverance of the *Protestants* of that Kingdom from the BLOODY MASSACRE begun by the *Irish Papists*, on the 23d of *October*, 1641. By JOHN RAMSEY, Rector of *Langdon* in *Kent*.

*And shortly will be published,*

A large *Appendix* to this Piece of Mr. HUYGENS on the *Value of Chances*; to be printed and sold single, in order to be bound up with it.

[1]

THE  
**V A L U E**  
 OF  
*C H A N C E S.*

**A**LTHOUGH in Games depending entirely upon Fortune, the Success is always uncertain; yet it may be exactly determin'd at the same time, how much more likely one is to win than lose. As, if any one shou'd lay that he wou'd throw the Number *Six* with a single Die the first throw, it is indeed uncertain whether he will win or lose; but how much more probability there is that he shou'd lose than win, is presently determin'd, and easily calculated. So likewise, if I agree with another to play the first Three Games for a certain Stake, and I have won one of my Three, it is yet uncertain which of us shall first get his third Game; but the Value of my Expectation and his likewise may be exactly discover'd; and consequently it may be determin'd, if we shou'd both agree to give over play, and leave the remaining Games unfinish'd, how much more of the Stake comes to my Share than his; [2] or, if another desired to purchase my Place and Chance, how much I might just sell it for. And from hence an infinite Number of Questions may arise between two, three, four, or more Gamesters: The satisfying of which being a thing neither vulgar nor useless, I shall here demonstrate in few Words, the Method of doing it; and then likewise explain particularly the Chances that belong more properly to Dice.

P O S T U L A T

As a Foundation to the following *Proposition*, I shall take Leave to lay down this Self-evident Truth: *That my Chance or Expectation to win any thing is worth just such a Sum, as wou'd procure me in the same Chance and Expectation at a fair Lay.* As for Example, if any one shou'd put 3 Shillings in one Hand, without letting me know which, and 7 in the other, and give me Choice of either of them; I say, it is the same thing as if he shou'd give me 5 Shillings; because with 5 Shillings I can, at a fair Lay, procure the same even Chance or Expectation to win 3 or 7 Shillings.

## P R O P. I.

*If I expect a or b, and have an equal Chance of gaining either of them, my Expectation is worth  $\frac{a+b}{2}$ .*

[3] To trace this Rule from its first Foundation, as well as demonstrate it, having put  $x$  for the value of my Expectation, I must with  $x$  be able to procure the same Expectation at a fair Lay. Suppose then that I play with another upon this Condition, That each shall stake  $x$ , and he that wins give the Loser  $a$ . 'Tis plain, the Play is fair, and that I have upon this Agreement an even Chance to gain  $a$ , if I lose the Game; or  $2x - a$ , if I win it: for I then have the whole stake  $2x$ , out of which I am to pay my Adversary  $a$ . And if  $2x - a$  be supposed equal to  $b$ , then I have an even Chance to gain either  $a$  or  $b$ . Therefore putting  $2x - a = b$ , we have  $x = \frac{a+b}{2}$ , for the Value of my Expectation. *Q.E.I.*

THE Demonstration of which is very easy: For having  $\frac{a+b}{2}$ , I can play with another, who shall likewise stake  $\frac{a+b}{2}$ , upon Condition that the Winner shall pay the Loser  $a$ . By which means I must necessarily have an equal Expectation to gain  $a$ , if I am Loser, or  $b$ , if I am Winner; for then I win  $\frac{a+b}{2}$ , the Whole Stake, out of which I am to pay the Loser  $a$ . *Q.E.D.*

IN Numbers. If I have an equal Chance to 3 or 7, then my Expectation is, by this *Proposition*, worth 5, and it is certain I can with 5, again procure the same Expectation: For if Two of us stake 5 a piece upon this Condition, That he that wins pay the other 3, 'tis plain the Lay is just [4] and that I have an even Chance to come off with 3, if I lose, or 7 if I win; for then I gain 10, and pay my Adversary 3 out of it. *Q.E.D.*

## P R O P. II.

*If I expect a, b, or c, and each of them be equally likely to fall to my Share, my Expectation is worth  $\frac{a+b+c}{3}$ .*

To calculate which, I again put  $x$  for the value of my Expectation: Therefore having  $x$ , I must be able, by fair Gaming, to procure the same Expectation. Supposing then I play with two others upon this Condition, That every one of us stake  $x$ ; and I agree with one of them, that which soever of us Two wins, shall give the loser  $b$ ; and with the other, that which soever of us Two wins, shall give the Loser  $c$ . It appears evidently, that the Lay is very fair, and that I have by this means an equal Chance to gain  $b$ , if the first wins; or  $c$  if the second wins; or  $3x - b - c$ , if I win my self; for then I have the whole Stake  $3x$ , but of which I give  $b$  to one, and  $c$  to the other. But if  $3x - b - c$  be supposed equal to  $a$ , then I have equal Expectation of

$a$ ,  $b$ , or  $c$ , Therefore putting  $3x - b - c = a$ , we shall find  $x = \frac{a+b+c}{3}$ , for the value of my Expectation. *Q.E.I.*

AFTER the same Manner, an even Chance to  $a$ ,  $b$ ,  $c$  or  $d$ , will be found worth  $\frac{a+b+c+d}{4}$ . And so on.

[5]

### P R O P. III.

*If the number of Chances I have to gain  $a$ , be  $p$ , and the number of Chances I have to gain  $b$ , be  $q$ , supposing the Chances equal; my Expectation will then be worth  $\frac{ap+bq}{p+q}$ .*

To investigate this Rule, I again put  $x$  for the value of my Expectation which must consequently procure me the same Expectation in fair Gaming. I take therefore such a Number of Gamesters as may, including my self, be equal to  $p + q$ , every one of which stakes  $x$ ; so that the whole stake is  $px + qx$ , and all play with an equal Expectation of winning. With so many Gamesters as are expressed by the Number  $q$ , I agree singly, that whoever of them wins, shall give me  $b$ ; and if I win, he shall have  $b$  of me: And with the rest, express'd by  $p - 1$ , I singly make this Agreement, That whoever of them wins, shall give me  $a$ ; and if I win, he shall receive  $a$  of me. It is evident, our playing upon this Condition is fair, no Body having any injury done him; and that my Expectation of  $b$  is  $q$ ; my Expectation of  $a$  is  $p - 1$ ; and my Expectation of  $px + qx - bq - ap + a$  (*i.e.* of winning) is 1: for then I gain the whole Stake  $px + qx$ , out of which I must pay  $b$ , to every one of the Gamesters  $q$ , and  $a$  to every one of the Gamesters  $p - 1$ , which together makes  $bq + ap - a$ . If therefore  $px + qx - bq - ap + a$  be equal to  $a$ , I shou'd have  $p$  Expectations of  $a$  (for I had  $p - 1$  Expectations of  $a$ , and [6] 1 Expectation of  $px + qx - bq - ap \leftarrow a$ , which is now supposed equal to  $a$ ;) and  $q$  Expectations of  $b$ ; and consequently am again come to my first Expectation. Therefore  $px + qx - bq - ap + a = a$ , and consequently  $x = \frac{ap+bq}{p+q}$ , is the value of my Expectation. *Q.E.I.*

IN Numbers. If I have 3 Expectations of 13 and 2 Expectations of 8, the value of my Expectations wou'd by this Rule be 11. And it is easy to show, that having 11, I cou'd again come to the same Expectations. For playing against Four more, and every one of us staking 11; with Two of them I agree singly, that he that wins shall give me 8; or to give him 8, if I win: And with the other Two in like manner, that which soever wins, shall give me 13; or give him so much if I win. The Play is manifestly fair, and I have just 2 Expectations of 8, if either of the Two that promis'd me 8 shou'd win; and 3 Expectations of 13, if either of the Two that are to pay me 13 shou'd win, or if I win my self, for then I gain the whole Stake, which is 55; from which,

deducting 13, a piece for the last Two I bargain'd with, and 8 a piece for the other Two, there remain 13 for my self. *Q.E.D.*

[7]

P R O P. IV.

*To come to the Question first propos'd, How to make a fair Distribution of the Stake among the several Gamesters, whose Chances are unequal? The best way will be to begin with the most easy Cases of that Kind.*

SUPPOSING therefore that I play with another upon this Condition, That he who gets the first three Games shall have the Stake; and that I have won two of the three, and he only one. I desire to know, if we agree to leave off and divide the Stake, how much falls to my Share?

IN the first place we must consider the number of Games still wanting to either Party: For it is plain, that supposing we had agreed the Stake shou'd be deliver'd to him that shou'd win the first twenty Games, and I had won nineteen of them, and the other only eighteen; my Chances wou'd have then been just so much better than his, as it is in the present Case, where I am supposed to have won two out of the three, and he only one: because in both Cases there remains but one Game for me to win and two for him.

MOREOVER to find how to share the Stake, we must have regard to what would happen, if both play'd on: For it is manifest, that if I win the next Game, my Number is compleated, and the Stake, which call  $a$ , is mine. But if the other gets the next Game, then both our Chances will be even, because we want but one Game apiece, and each of them [8] worth  $\frac{1}{2}a$ . But it is plain, I have an equal Chance to win or lose the next Game; and consequently an equal Chance to gain  $a$ , or  $\frac{1}{2}a$ ; which by *Prop.* 1. is worth half the Sum of them both, *i.e.*  $\frac{3}{4}a$ . *Q.E.I.*

My Playfellow's Share, which of course must be the remaining  $\frac{1}{4}a$ , might be first found after the same manner. From whence it appears, That he who would play in my room, ought to give me  $\frac{3}{4}a$  for my Chance; and consequently that whoever undertakes to win one Game, before another shall win two, may lay 3 to 1 Odds.

P R O P. V.

*Suppose I want one Game of being up, and my Adversary wants three; How must the Stakes be divided?*

LET us again consider what wou'd be the Consequence, if I shou'd get the next Game; 'tis plain I should win the Stake, suppose  $a$ ; but if the other shou'd get it, he wou'd still want two Games and I but one; and consequently

our Case would be the same with that mention'd in the foregoing Proposition and my Share,  $\frac{3}{4}a$ , as is there demonstrated. [9] Therefore, since I have an equal Chance to gain  $a$ , or  $\frac{3}{4}a$ , my Expectation must by *Prop.* 1. be worth  $\frac{7}{8}a$ ; and my Adversary's Share, the remaining  $\frac{1}{8}a$ . So that my Chance is to his, as 7 to 1. *Q.E.I.*

AND as the *Solution* of the foregoing Case is necessary to solving this last, so is the *Solution* of this last necessary to solving the following one, where I am supposed to want but one Game, and my Adversary four; for then my Share will be found, after the same manner, to be  $\frac{15}{16}$  of the Stake, and his,  $\frac{1}{16}$ .

#### P R O P. VI.

*Suppose I have two Games to get, and my Adversary three.*

THEREFORE after the next Game, I shall want but one more, and he three (in which Case my Share, by the foregoing *Prop.* is  $\frac{7}{8}a$ ) or we shall want two a piece, and then my Share is  $\frac{1}{2}a$ , both our Chances being equal. But I have an even Chance to win or lose the next Game, and consequently have an equal Expectation of obtaining  $\frac{7}{8}a$ , or  $\frac{1}{2}a$ , wch [10] by *Prop.* 1 is worth  $\frac{11}{16}a$ . So that eleven Parts of the Stake fall to my Share, and five to his. *Q.E.I.*

#### P R O P. VII.

*Suppose I want two Games, and my Adversary four.*

THEREFORE it will either fall out, that by winning the next Game, I shall want but one more, and he four, or by losing it I shall want two, and he shall want three. So that by *Schol. Prop.* 5. and *Prop.* 6., I shall have an equal Chance for  $\frac{15}{16}a$  or  $\frac{11}{16}a$ , which, by *Prop.* 1 is just worth  $\frac{13}{16}a$ . *Q.E.I.*

#### C O R O L L.

FROM whence it appears, that he that is to get two Games, before another shall get four, has a better Chance than he is to get one, before another gets two Games. For in this last Case, namely of 1 to 2 his Share by *Prop.* 4 is but  $\frac{3}{4}a$ , which is less than  $\frac{13}{16}a$

#### P R O P. VIII.

*Suppose now there are three Gamesters and that the first and second want a game a piece, and the third wants two Games.*

[11] To find therefore the Share of the first Gamester, we must again examine what he wou'd gain, if either he himself, or one of the other two gets

the next Game: If he gets it, he wins the whole Stake  $a$ ; if the second gets it, because he likewise wanted but one Game, he has the Stake, and the first gets 0; and if the third gets it, then they will all three want one Game a piece, and the Share of each of them will consequently be  $\frac{1}{3}a$ . The first Gamester therefore has an equal Expectation of gaining  $a$ , or 0, or  $\frac{1}{3}a$ , (since each has the same likelihood of winning the next Game,) which, by *Prop. 2.* is worth  $\frac{4}{9}a$ . The Share of the second will be likewise  $\frac{4}{9}a$ , and there will be  $\frac{1}{9}a$  remaining for the third; whose Share might, after the same manner, be found separately from the others, and theirs determin'd by that. *Q.E.I.*

P R O P. IX.

*To find the several Shares of as many Gamesters, as we please, some of which shall want more Games, others fewer; we must consider what he, whose Share we want to find, wou'd gain, if he, or any one of the others wins the next Game: Then adding together what he wou'd gain in all those particular Cases, and dividing the sum by the Number of Gamesters, the Quotient gives the particular Share required.*

[12] SUPPOSE, for Example, there were three Gamesters  $A$ ,  $B$ , and  $C$ , and  $A$  wanted one Game of being up, and  $B$  and  $C$  wanted two apiece, and I desire to find the Share that  $B$  has in the Stake  $q$ .

FIRST of all we must see what will happen to  $B$ , if either he, or  $A$ , or  $C$  wins the following Game.

IF  $A$  wins it, he is up and consequently  $B$  gets 0. If  $B$  himself wins it, he and  $A$  will still want a Game apiece, and  $C$  will want two; and consequently  $B$ , by *Prop. 8.* gets  $\frac{4}{9}q$ . But lastly, if  $C$  wins it, then  $A$  and  $C$  will still want one Game apiece, and  $B$  will want two; and consequently  $B$ , by *Prop. 8.* will in this Case get  $\frac{1}{9}q$ .

Now the three several Gains of  $B$  in all these particular Cases, which are 0,  $\frac{4}{9}q$ , and  $\frac{1}{9}q$ , added together make  $\frac{5}{9}q$ ; which Sum being divided by 3, the number of Gamesters, gives  $\frac{5}{27}q$  for the Share of  $B$ . *Q.E.I.*

THE Demonstration of this is plain from *Prop. 2.* For since  $B$  has an equal Chance to 0,  $\frac{4}{9}q$  or  $\frac{1}{9}q$ , his Expectation is, by that *Prop.* worth  $\frac{0+\frac{4}{9}q+\frac{1}{9}q}{3}$  *i.e.*  $\frac{5}{27}q$ . And 'tis evident, this Divisor 3, is the Number of Gamesters. *Q.E.D.*

[13] BUT in order to find what any one will gain in every particular Case, supposing himself or any of the rest should win the next Game; the more simple and intermediate Cases must first be investigated. For as this last Case cou'd not have been resolv'd without that calculated in *Prop. 8.* where the Games wanting were 1,1,2; so likewise every single Person's Share, in case the Games wanting were 1,2,3, cannot be found, without first determining the Case where the Games wanting are 1,2,2 (which is already done in *Prop.*

9. and that likewise where the Games wanting are 1,1,3, which may, by *Prop.* 8. be easily calculated. And after this manner may be resolv'd all the Cases comprehended in the following Table, and others, *ad Infinitum*.

[14]

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TABLE for Three Gamesters

<i>Games wanting Shares.</i>	$\frac{1. 1. 2.}{4. 4. 1.}$ 9.	$\frac{1. 2. 2.}{17. 5. 5.}$ 27.	$\frac{1. 1. 3.}{13. 13. 1.}$ 27.	$\frac{1. 2. 3.}{19. 6. 2.}$ 27.
<i>Games wanting Shares.</i>	$\frac{1. 1. 4.}{40. 40. 1.}$ 81.	$\frac{1. 1. 5.}{121. 121. 1.}$ 243.	$\frac{1. 2. 4.}{178. 58. 7.}$ 243.	$\frac{1. 2. 5.}{542. 179. 8.}$ 729.
<i>Games wanting Shares.</i>	$\frac{1. 3. 3.}{65. 8. 8.}$ 81.	$\frac{1. 3. 4.}{616. 82. 31.}$ 729.	$\frac{1. 3. 5.}{629. 87. 13.}$ 729.	
<i>Games wanting Shares.</i>	$\frac{2. 2. 3.}{34. 34. 13.}$ 81.	$\frac{2. 2. 4.}{338. 338. 53.}$ 729.	$\frac{2. 2. 5.}{353. 353. 23.}$ 729.	$\frac{2. 3. 3.}{133. 55. 55.}$ 243.
[Continuation of same line in original]			$\frac{2. 3. 4.}{451. 195. 83.}$ 729.	$\frac{2. 3. 5.}{1433. 635. 119.}$ 2187.

[15] As to what belongs to DICE, the Questions propos'd concerning them are, In how many Times we may venture to throw *Six*, or any other Number, with a single Die, or two *Sixes* with two Dice, or three *Sixes* with three Dice; and such-like.

To resolve which, we must observe, First, That there are six several Throws upon one Die, which all have an equal probability of coming up. That upon two Dice there are 36 several Throws, equally liable to be thrown, for any one of the six Throws of one Die may come up with every one of the six Throws of the other; and so 6 times 6 will make 36 Throws. So likewise, that three Dice have 216 several Throws; for the 36 Throws on the two Dice may happen together with any one of the 6 Throws of the third Die; and so 6 times 36 will make 216 Throws. After the same manner, 'tis plain, four dice will have 6 times 216, *i.e.* 1296 Throws; and so on, may we calculate

the Throws upon any Number of Dice, taking always at the addition of every Die 6 times the preceding Number of Throws.

It is farther to be observ'd, that upon two Dice there is only one Throw that can produce 2 or 12, and two Throws that can produce 3 or 11. For if we call the Dice *A* and *B*, 'tis plain, the throw 3 may be made up of the 1 of *A* and 2 of *B*; or of the 1 of *B* and 2 of *A*. So likewise will 11 be produced by the 5 of *A*, and 6 of *B*; or by the 5 of *B*, and 6 of *A*. The Number 4 may be thrown three Ways, by 1 of *A* and 3 of *B*; or 3 of *A* and 1 of *B*; or 2 of *A* and 2 of *B*.

[16] X may likewise be thrown three several Ways.

V or IX has 4 several Throws.

VI or VIII has 5 Throws.

VII has 6 Throws.

$$\begin{array}{c} \text{Upon 3 Dice,} \\ \text{the Numbers} \end{array} \left\{ \begin{array}{c} 3 \text{ or } 18 \\ 4 \text{ or } 17 \\ 5 \text{ or } 16 \\ 6 \text{ or } 15 \\ 7 \text{ or } 14 \\ 8 \text{ or } 13 \\ 9 \text{ or } 12 \\ 10 \text{ or } 11 \end{array} \right\} \begin{array}{c} \text{may come up by} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \left\{ \begin{array}{c} 1 \\ 3 \\ 6 \\ 10 \\ 15 \\ 21 \\ 25 \\ 27 \end{array} \right\} \begin{array}{c} \text{several} \\ \text{Throws} \end{array}$$

#### PROP. X.

*To find how many Throws one may undertake to throw the Number 6 with a single Die.*

If any one wou'd venture to throw *Six* the first Throw, 'tis plain there is but 1 Chance by which he might win the Stake; and 5 Chances by which he might lose it: For there are 5 Throws against him and only 1 for him. Let the stake be called *a*. Since therefore he has one Expectation of *a*, and 5 Expectations of 0, his Chance is, by *Prop. 2.* worth  $\frac{1}{6}a$ ; and consequently there remains to his Adversary  $\frac{5}{6}a$ . So that he that wou'd undertake to throw *Six* the first Throws, must lay only 1 to 5.

[17] HE that wou'd venture to throw *Six* once in two Throws, may calculate his Chance after the following manner: If he throws *Six* the first Throw, he gains *a*; if the contrary happens, he has still another Throw remaining, which by the foregoing Case, is worth  $\frac{1}{6}a$ . But he has only 1 Chance to throw *Six*

the first Throw and 5 Chances to for the contrary: Therefore before he throws, he has one Chance for  $a$  and five Chances for  $\frac{1}{6}a$ , the Value of which, by *Prop.* 3. is  $\frac{11}{36}a$ . And consequently there remains  $\frac{25}{36}a$ , to the other that lays with him. So that their several Chances, or the Values of their several Expectations, bear the Proportion of 11 to 25, *i.e.* less than 1 to 2.

HENCE, after the same Method, the Chance of him who wou'd venture to throw *Six* one in three Throws, may be investigated and found worth  $\frac{91}{216}a$ , so that he may lay 91 against 125; which is a little less than 3 to 4.

HE who undertakes to throw it once in four Times, has a Chance worth  $\frac{671}{1296}a$ , and may lay 671 to 625, *i.e.* something more than 1 to 1.

HE who undertakes to throw it once in five Times, has a Chance worth  $\frac{4651}{7776}a$ , and may lay 4651 against 3125, *i.e.* something less than 3 to 2.

[18] HE who undertakes the same in 6 Throws has a Chance worth  $\frac{31031}{46656}a$ , and may lay 31031 to 15625, *i.e.* a little less than 2 to 1.

AND thus the *Problem* be resolv'd in what Number of Throws we please. *Q.E.I.*

BUT it is possible for us to proceed after a more compendious Method, as shall be shown in the following *Proposition*, without which the Calculation wou'd otherwise be much more prolix.

#### PROP. XI.

*To find in how many Throws one may venture to throw the Number 12 with two Dice.*

IF any one shou'd pretend to throw 12 the first throw, 'tis plain he has but one Chance of winning, *i.e.* of gaining  $a$ ; and 35 Chances of losing, or gaining 0, because there are in all, 36 several Throws. And consequently his Expectation, by *Prop.* 3. is worth  $\frac{1}{36}a$ .

HE that undertakes to do it in two Throws, if it comes up in the first Throw, will obtain  $a$ ; and if it does not he has yet one Throw to come, which, by what has been said before, is worth  $\frac{1}{36}a$ . But there is only one Chance for throwing 12 the first Throw and 35 Chances against it: Therefore since he has 1 Expectation of  $a$  and 35 of  $\frac{1}{36}a$ , his [19] Chance by *Prop.* 3. is worth  $\frac{71}{1296}a$ ; and his Adversary's the remaining  $\frac{1225}{1296}a$

FROM these Two Cases we can determine the value of his Chance who ventures to do the same in Four Throws, without considering the Chance of him that undertakes to do it in Three.

FOR he that ventures to throw 12 once in four times throwing, if he does not throw it the first or second time, gains  $a$ ; and if the contrary happens, he has still two Throws more, which, by what has been said before, are worth  $\frac{71}{1296}a$ : For which Reason likewise, he must have 71 Chances for throwing

12 in one of the two first, and 1225 Chances against it. Therefore at his beginning to throw he has 71 Expectations of  $a$ , and 1225 Expectations of  $\frac{71}{1296}a$ , which, by *Prop. 2.*, are worth  $\frac{178991}{1679616}a$ : And the value of his Chance that plays against him will be the remaining  $\frac{1500625}{1679616}a$ . Which shows that their Chances are to one another, as 178991 to 1500625.

FROM which likewise, without calculating any other Cases for that Purpose, may be found by the same Way of Reasoning, the worth of his Expectation who undertakes to throw two *Sixes* once in 8 Throws. And from thence the worth of his Expectation [20] who ventures to do the same in 16 Throws. And from the Expectation of this last being found, together with his also who ventures it in 8 Throws, may be determined the value of his Expectation who undertakes it in 24 Throws. In which Operation, because the principal Question is, In what Number of Throws one may lay an even Wager to throw two *Sixes*; we may cut off some of the hindermost Figures from the long Numbers that arise in the midst of the Calculations, and which wou'd otherwise encrease prodigiously. And by this Means, I find he that undertakes it in 24 Throws wants something of an even Chance to win; and he that lays to do it in 25 Throws, has something the better side of the Wager.

#### PROP. XII.

*To find with how many Dice one may undertake to throw two Sixes the first Throw.*

THIS is the same thing as if we wou'd know, in how many Throws one may undertake to throw two *Sixes* with a single Die. He that ventures to do this in two Throws, has, by what has been demonstrated in *Prop. 11.* a Chance worth  $\frac{1}{36}a$ . He that undertakes it in three Throws, if he does not happen to throw one *Six* the first Throw, has yet two more to come, which as before, are worth  $\frac{1}{36}a$ . But if the first Throw chance to be a *Six*, he has two Throws more to throw one *Six*, which by *Prop. 10.* are worth  $\frac{11}{36}a$ . [21] Now 'tis plain that there is one Chance for throwing a *Six* the first Throw, and five Chances to the contrary: So that before he throws, he has one Chance for  $\frac{11}{36}a$ , 5 Chances of  $\frac{1}{36}a$ , which by *Prop. 3.*, are worth  $\frac{16}{216}a$ , or  $\frac{2}{27}a$ . After this manner, by continually taking one more throw, we find that we may in 10 Throws with one Die, or in one Throw with 10 Dice, undertake to throw two *Sixes*, and that with Advantage.

## PROP. XIII.

*Supposing I lay with another to take one Throw with a pair of Dice upon these Terms, That if the Number 7 comes up, I shall win, and if 10 comes up, he shall win; and after this Bargain made, we consent to draw Stakes by a fair Division, according to the Value of our Chances in the present Contract: To find what shall be our several Shares.*

BECAUSE, of the 36 Throws upon two Dice, there are but 6 which consist of the Number 7 a piece, and 3 which consist of the Number 10 a piece; there remain 27 Throws, which, if any one of them chance to come up, will make us neither win nor lose, and consequently entitle each of us to  $\frac{1}{2}a$ . But if none of those Throws shou'd happen, I have 6 Chances for winning  $a$ , or and Chances of losing, or having 0, which by *Prop. 3.* is as good as if [22] I had  $\frac{2}{3}a$ . Therefore I have from the beginning 27 Chances for  $\frac{1}{2}a$  and 9 Chances for  $\frac{2}{3}a$ , which by *Prop. 3.* is worth  $\frac{13}{24}a$ ; and there remains to him that plays against me  $\frac{11}{24}a$ .

## PROP. XIV.

*If my self and another play by turns with a pair of Dice upon these Terms, That I shall win if I throw the Number 7, or he if he throws 6 soonest, and he to have the Advantage of first Throw: To find the Proportion of our Chances.*

SUPPOSE my Chance worth  $x$ , and call the Stake  $a$ ; therefore his Chance will be  $= a - x$ . 'Tis plain then, that whenever it comes to his Turn to throw, my Chance ought to be  $= x$ . But when it is my Turn to throw, my Chance must be worth something more. Let its Value, then, be express'd by  $y$ . Now because of the 36 Throws upon a pair of Dice, 5 are made up of the Number 6 apiece, and may make my Adversary win, and 31 of them are against him, *i.e.* promote my Turn of throwing; I have, before he begins to throw, 5 Chances of obtaining 0, and 31 Chances of obtaining  $y$ , which, by *Prop. 3.* are worth  $\frac{31}{36}y$ . But my Chance from the beginning was supposed worth  $x$ , and therefore  $\frac{31}{36}y = x$ , and consequently [23]  $y = \frac{36x}{31}$ . It was further supposed that in my Turns of throwing, my Chance was worth  $y$ . But when I'm to throw, I have 6 Chances of gaining  $a$ , because there are 6 Throws of the Number 7 apiece, which wou'd give me the Game; and 30 Chances, which will bring it to my Adversary's Turn to throw, *i.e.* make me gain  $x$ ; which, by *Prop. 3.* are worth  $\frac{6a+30x}{36}$ . And because this  $= y$ , which was before found  $= \frac{36x}{31}$ ; therefore  $\frac{6a+30x}{36} = \frac{36x}{31}$ . From which Equation will be had  $x = \frac{31}{61}a$ , the Value of my Chance. And by consequence my Adversary's will be worth  $\frac{30}{61}a$ , so that the Proportion of my Chance to his is, as 31 to 30.

*For a concluding Ornament to this Work, we shall subjoin the following Problems.*

PROBLEM I.

*A and B play together with a pair of Dice upon this Condition, That A shall win if he throws 6, and B if he throws 7; and A is to take one Throw first, and then B two Throws together, then A to take two Throws together, and so on both of them the same, till one wins. The Question is, What Proportion their Chances bear to one another? Answ. As 10355 to 12276.*

[24]

PROBLEM II.

*THREE Gamesters, A, B, and C, taking 12 Counters, 4 of which are white, and 8 black, play upon these Terms: That the first of them that shall blindfold choose a white Counter shall win; and A shall have the first Choice, B the second, and C the third; and then A to begin again, and so on in their Turns. What is the Proportion of their Chances?*

PROBLEM III.

*A lays with B, that out of 40 Cards, i.e. 10 of each different Sort, he will draw 4, so as to have one of every Sort. And the Proportion of his Chance to that of B, is found to be as 1000 to 8139.*

PROBLEM IV.

*HAVING chosen 12 Counters as before, 8 black and 4 white, A lays with B that he will blindfold take 7 out of them, among which there shall be 3 black ones. Quære, What is the Proportion of their Chances?*

PROBLEM V.

*A and B taking 12 Pieces of Money each, play with 3 Dice on this Condition, That if the Number 11 is thrown, A shall give B one Piece, but if 14 be thrown, then B shall give one to A; and he shall win the Game that first gets all the Pieces of Money. And the Proportion of A's Chance to B's is found to be, as 244,140,625 to 282,429,536,481.*

*F I N I S.*